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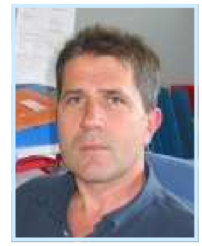
In this issue: A new initiative. Mark Fackrell's article explaining the mystery of exponential and phase-type modelling, summarised here, is mailed separately and on the web. Readers responses to old and new physics lead to concepts based on structure, pattern and process. Martin Pitt reports the success of the 33rd ORAHS conference in St. Etienne. We have leads to several journal articles. HSCM Portrush 18 - 20 March 2008. First congratulations to Thierry.

Professor Thierry Chaussalet



Thierry's research and scholarship, as Leader of the Health and Social Care Modelling Group at the University of Westminster, has now been recognised by the University as a full professor. Also his research, with Haifeng Xie, into modelling the committed cost of currently funded clients in residential and nursing home care, is being prototyped by the Department of Health for use by English Social Service Departments. Success heralds a break through in understanding the benefits of modelling the process and cost of care.

Phase-type Distributions in Healthcare Modelling I : Mark Fackrell

Necessity is the mother of invention. Mathematics is a high level language, with many symbols expressing concepts. Till now we have scanned formulae, however, given the quality of Mark's article and the promise of two more, we have created a Tutorial section on the Nosokinetics website at www.nosokinetics.org.

This article first explains why *stochastic* (random) models are needed. One compelling reason is that performance measures need to be determined, so sensible decisions can be made. Another is that stochastic models often perform better than deterministic models, because randomness underlies many real world systems.

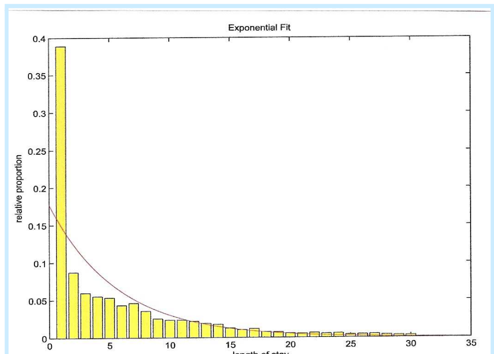


Figure 1. Histogram of length of stay of 4696 patients transferred to the Royal Melbourne hospital. Showing that the fit with a single exponential (λ = 0.1779, average stay 5.6 days) is not very good.

Probability theory relies on the concept of a *random variable.* Consider tossing a coin. The experiment may be tossing three fair coins, and the random variable is the "number of heads". Associated with any random variable is a *distribution function.* In this example the probability of tossing 0,1,or 2 heads is 7/8.

Discrete random variables have discrete values e.g. 0,1,2 and 3. Whereas, *continuous* random variables take values from an interval e.g. the "length of stay in an emergency department". One very important continuous distribution is the *exponential* distribution, which has the unique feature of being *memoryless*.

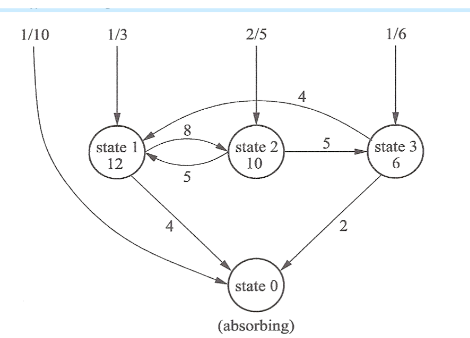


Figure 2. State transition diagram of a 4-state continuous-time Markov chain with one absorbing state.

Figure 1 shows that the fit between a single exponential and length of stay of transferred patients is not very good. So better stochastic models are needed. Here is where *phase type* models come into the picture. Here we need the concept of a *(finite state) continuous time Markov chain.*

In Figure 2 the *state space S* = {*0,1,2,3*} and we select a state according to the *initial state probability vector* = (*1/10, 1/3, 2/5, 1/6*). Once in each state we spend an exponentially distributed time λ in each state (for example *λ =12* for state 1) and then move to another state with a certain rate. We go from state 1 to state 2 with rate 8, and from state 1 to state 0 (the absorbing state) with rate 4. So rate 12 can be thought of as the rate "out" of state 1. The process continues until we end up in state 0 where we stop.

Read the full text at http://www.nosokinetics.org/tutorial.htm.

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One size does not fit all: structure, pattern, process and location play their part

By Peter Millard

Following my contribution on 'Old Physics' in the last news letter my eyes opened. I've been living in the past. The future is different. Reading Fritjof Capra's 1996 book, The Web of Life. A new synthesis of mind and matter [1], I realised that a whole new dimension -autopoietic networks - self bounded, self-generating and self perpetuating systems - has to be grasped to begin to understand the reality of clinical care.

Living in the past

First Elia El-Darzi from the Harrow Campus of Westminster University said he had discussed Ouspensky's comparison between Old and New Physics (see NKNews4.3) with a colleague who said the new physics is "outdated". "That's not the point" I replied. "The analogy with 'old physics' highlights the problem, it's not about new physics". But now I realise it is, and it's about biology, systems theory and many other aspects as well.

Loy Lobo from BT Health emailed and suggested that I should look at System Dynamics. On the Systems Dynamics Society web site, I came across an article written in 1991 [2] by Jay W Forrester, which quotes Skinner, making an analogy between the knowledge of the Greeks and the modern world.

> *"Physics and biology have come a long way, but there has been no comparable development of anything like a science of human behavior... Aristotle could not have understood a page of modern physics or biology, but Socrates and his friends would have little trouble in following most current discussions of human affairs."* [3]

System Dynamics Society - http://www.systemdynamics.org/

Sustainability Institute - http://www.sustainabilityinstitute.org/ (look at the project on Diabetes and Chronic Disease).

Strategy Dynamics - http://www.strategydynamics.com/
The site provides some examples of Microworlds.

Bridging the gap

Reading further, my thoughts turned to the 1980's when we started a BSc module and to the workshop at the 2005 Craiova conference hosted by Florin Gorunescu. Then I realised taking a business analogy "We have to know how and why the elephant dances."

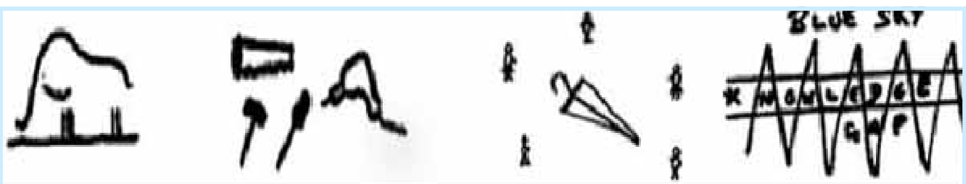

Figure 1. Bridging the gap. Everyone sees the world from the point of view on which they stand. There are many opinions and many tools. Until politicians see the need for sustainable systems, tools gather dust in the depths of academe.

Innovation: Collaboration

When we started the BSc Module on Ageing the students knowledge of basic anatomy was the same as mine, but their knowledge of the mechanisms of movement, physiology, biochemistry, molecular biology, genetics, psychology was streets ahead of mine. To solve this problem, collaboratively, the professors agreed to give the lectures on ageing, while I sat in the back and made notes. What surprised them, and me, was that they too had to learn about the ageing aspects of their subject, for the impact of the ageing on structure and function was not part of their syllabus. Thus a new course began, which ended when the Research Evaluation Exercise began and colleagues were told to concentrate on research. (cont. p3).

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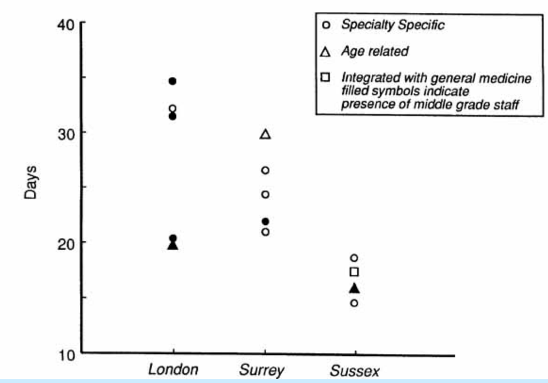
Innovation: structure

In 1948, when the NHS began, responsibility for providing long term institutional care was taken from local government and placed on the Regional Hospital Boards. From that unlikely beginning the specialty of geriatrics began. Evidence of lack of medical leadership and rehabilitation motivated the change. All services had to be developed. The aim was to solve the problem of 'bed-blocking' in acute hospitals.

Innovation: pattern

By the 1980's three styles of practice -needs related, age related and integrated - had emerged. Dependent on the local facilities, the aptitudes of the consultants, relations with colleagues and access to beds in acute hospitals.

Fig 6.2 Relationship between the calculated group one expected stay in daysin geriatric medicine and the location of the Health District



Innovation: process

In 1991, Amy Roberts, a young American doctor astutely observed that Sussex departments of geriatric medicine treated patients faster than those in Surrey, and Surrey was faster than London. See figure 1. Note that the differences are independent of the pattern of clinical practice.

The research involved visiting the hospitals, interviewing medical and nursing staff, observing the facilities and collecting census data to model the flow. Thousands of facts were collected, but only three related to ease of discharge of long stay patients and poverty of rehabilitation explained these differences. (see Millard's PhD thesis at http://www.nosokinetics.org/publications.htm)

Crashing the system

During the 1980's and early 1990's government fostered a entrepreneurial boom in voluntary and private services. Open access to public funds for poor people meant acute hospitals could bypass the geriatric medical services and discharge potential long-term patients directly to long-term nursing and residential care. When expenditure got out of control, social services were given responsibility for purchasing long-term. As geriatric medical services were no longer needed, physicians trained in geriatric medicine took on responsibility for acute care, and the physicians previously doing acute care moved over to expand organ based medical services. Now 'Bed-blocking' has returned and the way forward is unclear.

Light in the darkness

In June 2007, researchers reporting to UK Department of Health in "Modernising Adult Social Care - What's Working" [4] made the following observations:

> P106 – 107 "… (A) *rational linear approach to policy ... assumes policy 'levers' can be pulled and that action will follow without distortion until it reaches the 'front line'."*
>
> *"Policy does not come neatly tied up in sealed packages. It is made as people and organisations interpret it , translate it, try to make it meaningful within the frames of reference they bring to their work, and shape it in innovative ways. Indeed if this did not happen, innovation would not be possible."*
>
> *"In the case of social care … implementation relies on a large number of actors …. And rather than living and working in a simple organisational hierarchy, these actors are interdependent. They must negotiate action through a different and overlapping networks and sets of personal relations between staff and organisations."*

Conclusion

The world population is ageing. The cost of medical care is rising. The time is right for a world wide initiative to create a science base for the planning of health and social care. Which rejects 'levers' and recognises the symbiotic nature of co-operation in knowledge based systems.

1. Capra, Fritjof (1996) The web of life: a new synthesis of mind and matter. Harper Collins, London
2. Forrester, J.W. Systems Dynamics and the Lessons of 35 years. Chapter in the Systemic Basis od Policy Making ed Kenyon B.D. Greene. http://sysdyn.clexchange.org/sdep/papers/D-4224-4.pdf
3. Skinner, B. F. 1971. *Beyond Freedom and Dignity*. New York: Bantam Books. 215 pp
4. June 2007 Quote 2822179/ DH Publications Orderline dh@prolog.uk.com

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MASH News

Report from the 33rd International Conference on the Operational Research Applied to Health Services

By Martin Pitt

The ORAHS 33rd conference (see http://www.emse.fr/orahs) in St Etienne, France (15-20 July 2007) opened with more than 140 participants from over 20 countries - easily its biggest and most international gathering since its inception. This reflects the current growing interest in healthcare modelling and simulation across the world (which was also witnessed at the highly successful recent IMAHealth conference (see http://www.healthcareinformatics.org.uk/imahealth2007) in London in April 2007).

ORAHS is largely an academic conference with the vast majority of delegates drawn from research institutions. This year, however, there seemed to be a greater awareness of the need to address issues of implementation. A few presentations by participants from commercial or health service backgrounds were useful in highlighting the differing perspectives of other professional groups.

Over the five days of the conference, presentations ranged widely across many fields of Operational Research in health services. Workforce modelling, generic frameworks, hospital simulations, patient flow modelling, data mining, and policy issues were all well covered. A number of presentations addressed the critical issues of how to ensure more OR research finds its way into service implementation.

ORAHS places a strong emphasis on social and networking events. Plenty of time to sample the joys of French cuisine and visit the historic sites around St Etienne and Lyon. Time also for the delegates to meet and discuss potential collaborative projects, share ideas, and seek ways to extent their research. A key ingredient in the conference is the involvement of a significant contingent of post-graduate research students who have tutorial streams and dedicated events to support their involvement.

We will aim to include a fuller report from ORAHS 33 as well as detail from the many other recent health care modelling events (eg. the IMAHealth 2007 conference) in the next issue of the Nosokinetics News.

NIHR Programme Grants for Applied Research: Deadline 20 August 2007, 5:00pm

The funding scheme aims to

1. Provide evidence to improve health outcomes in England
2. Enable NHS trusts to tackle areas of high priority or need for health
3. Provide some stability of funding to support the long-term development of top quality applied research
4. Replace, in part, programmes supported by the Priorities and Needs component of NHS R&D

Annual budget of £75 million once it reaches full capacity. Individual awards maximum of £2m over a period of three to five years.

All NHS providers (including research general practices) in England may propose programmes, in collaboration with an appropriate academic partner or partners. Bids may be submitted by consortia including more than one NHS organisation. Bids may include support for patient/consumer groups leading or participating in programmes of research. It is possible that the academic partner could be a university outside England, if an appropriate case is made in the application. We would expect the application to make a strong case that the chosen academic partner was the organisation best placed to provide academic input to the planned research.

For further information see website: http://www.nihr-ccf.org.uk/site/programmes/programmegrants/default.cfm

The deadline for completion of a Registration of Intent to Submit is 5:00pm on 20 August 2007.

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What's in the journals?

Multi-stage model for whole hospital planning

Cochran JK, Bharti A. A multi-stage stochastic methodology for whole hospital bed planning under peak loading. *International Journal of Industrial and Systems Engineering* 2006;1(1/2):8-36.

A two stage process modelling a hospital with 400 beds. Flow diagrams show the pathways of patient care for the whole hospital and in medicine and surgery. A queuing network gives insight into optimal bed allocation at peak occupancy. And a system dynamic model shows the use of beds by department and to identifies services with too much or too little resources. Further work includes estimation of bed-blocking, staffing schedules and patient classification. The long term aim is real time integration of the models with the hospital database.

History teaches: 1976 stochastic model of bed distribution

Esogbue AO, Singk AJ. A stochastic model for an optimal priority bed distribution problem in a hospital ward. *Operations Research* 1976;24(5):884-898.

Develops a patient centred mathematical model of admission policies, a hybrid of traditional models, which seeks to maximise occupancy and minimize unsatisfied needs. The model provides a quantitative, rational and operational basis for establishing a priority "cut-off occupancy" that will maximise the medical benefits while differentiating for case types. Three types of cost are considered - Holding costs, Shortage costs and Fixed operating costs. A decision tree shows the possible patient outcomes - getting better, staying the same and getting worse. Use is demonstrated with data from a 1000 bed hospital in Cleveland.

Mathematical model of maternal services: why bother

Galväo RD, Espejo LGA, Boffey B. A hierarchical model for the location of perinatal facilities in the municipality of Rio de Janeiro. *European Journal of Operational Research* 2002;138: 495–517

With the aim of reducing perinatal mortality, a three level hierarchical model for the location of maternal and perinatal health care facilities in Rio was created. The mathematical solution using two basic heuristics was tested using 1995 data. The authors end with a warning "Why embark on a mathematical model if the Municipality are not going to use it?" — *I've got the answer to that. It's a step forward on the path. Also if they don't use it, someone else might.*

Predicting outcome of trauma patients

Clark, D. E., F. L. Lucas, et al. (2007). "Predicting hospital mortality, length of stay, and transfer to long-term care for injured patients." *J Trauma* 62(3): 592-600.

Data 369,829 patient records from the 1999 to 2993 National Trauma Data Bank. A multistate model, divided into four time periods, each with constant rates of death, discharge home and LTC transfer was used. Early mortality associated with severity of illness diminished with time. Age was a strong predictor of death and transfer: LTC transfer peaked at 6 to 9 days.

Controlled trials of different style stroke units – should all get good result?

Foley, N., K. Salter, et al. (2007). "Specialized stroke services: a meta-analysis comparing three models of care." *Cerebrovasc Dis* 23(2-3): 194-202.

Meta analysis of three styles of stroke units poses more questions than it answers. Maybe it just shows the strength of self-fulfilling prophecies. Five acute stroke units admitting patients up to 36 hours and staying for two weeks; four combining acute and rehabilitation and five post-acute rehabilitation units were all associated with significant reductions in mortality and dependency compared with their control group. Not surprisingly, post-acute units had significant reductions in mortality. Clearly the jury is still out. Pay your money and take your choice.

Transferred patients cost more

Golestanian, E., J. E. Scruggs, et al. (2007). "Effect of interhospital transfer on resource utilization and outcomes at a tertiary care referral center." *Crit Care Med* 35(6): 1470-6.

The observational cohort study of 4569 consecutive admissions to the intensive care unit in a large academic hospital showed transferred patients were sicker and had higher mortality and longer length of stay Stratified by disease severity and hospital mortality there was no difference between admitted and transferred patients in either ICU or hospital mortality. Risk stratification revealed that higher costs $9,600 (95% c.i. $6000-$13,400) was entirely confined to the longer stay of low risk transferred patients.

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Second International Health and Social Care Modelling Conference (HSCM 2008)

Portrush, Northern Ireland
18 - 20 March, 2008

Organised in association with the University of Ulster, School of Computing Information and Engineering at the Coleraine Campus, HSCM 2008 enables researchers and practitioners to meet in a convivial setting to, exchange ideas, examine the current modelling trends and issues, and develop new solutions and research directions to ultimately, improve patient and client care.

The conference venue is the Comfort Hotel at Portrush (see http://www.comforthotelportrush.com) , a small seaside town on the North Coast of Ireland, with beautiful beaches, convivial restaurants and friendly pubs. It is close to the Bushmills Distillery and Giant's Causeway and part of the Causeway Coast Area of Outstanding Natural Beauty.

For further details contact Sally McClean (si.mcclean@ulster.ac.uk).

Aims of HSCM 2008, Portrush, 18 - 20 March 2008

- Widening international understanding of the potential benefits of modelling
- Highlighting instances where theory and practice meet
- Encouraging close working relationships modellers and practitioners, and
- Increasing understanding of the different computational and data analytical methods used to measure and model health and social care services.

Modelling in Healthcare Stream of The Operational Research Society Conference

Edinburgh on 4th - 6th September

http://www.orsoc.org.uk/conference/papersubmission/conference_submit.asp?cid=14

Nosokinetics News is the newsletter of the UK Nosokinetics Group

Nosokinetics is the science / subject of measuring and modelling the dynamic aspects of patient and client movement (flow) through health and social care systems. From the Greek, literally, *noso* (disease) and *kinetics* (movement).

The group collaborates to organise conferences and disseminates news of our and others research and practical use of modelling to enhance decision making in health and social care systems. Our next International conference will be in Portrush, Northern Ireland in 18-20 March 2008.

To join or leave our JISC mailing list copy the link below and follow the instructions at

http://www.jiscmail.ac.uk/lists/NOSOKINETICS-NEWSLETTER.html

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Email address for submissions, letters - editor@nosokinetics.org

Phase-type Distributions in Healthcare Modelling I

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In the first of these three articles on phase-type (PH) distributions in healthcare I will briefly discuss the necessity of using stochastic models in healthcare, and introduce PH distributions. In the second article I will describe some of the ways in which PH distributions have already been used in healthcare modelling, and in the third, discuss some ways in which they could be further utilized. Fackrell [2] contains a more comprehensive treatment and bibliography.

Why do systems in the healthcare industry need modelling at all? One compelling reason is that performance measures need to be determined so that sensible decisions concerning the running of the systems can be made. For example, the manager of an emergency department needs to know, or at least have some idea of, how many presentations there will be in such-and-such a time interval so that an appropriate number of doctors and nurses can be rostered on. Mathematical models can provide powerful tools to analyse processes in healthcare (and, of course, in other industries as well).

The next question is: why model such processes *stochastically*? The simple answer is: because one does not know precisely what will happen in advance, but is likely to have some idea based on past experience. Often stochastic models perform better than deterministic ones because randomness underlies many real-world phenomena.

Probability theory relies on the concept of a *random variable*. There is a technical mathematical definition but for our purposes a random variable can be thought of as a measurement on the (random) outcome of an experiment. For example, the experiment may be tossing 3 fair coins, and the random variable the "number of heads". Associated with any random variable is a *distribution function* F, which is defined, for $x \in \mathbb{R}$ (that is, all real numbers x), as

$$F(x) \ = \ P(X \leq x).$$

In our coin tossing example $F(2) = P(X \leq 2) = \frac{7}{8}$ because the probability of getting 0, 1, or 2 heads in three tosses is $\frac{7}{8}$.

The random variable X is known as a *discrete* random variable because it takes on discrete values, namely 0, 1, 2, and 3. *Continuous* random variables take values from an interval. For example, the "length of stay in an emergency department" can be modelled by a continuous random variable because it takes on values that are greater than zero. Consequently, the distribution functions of such random variables are continuous functions for all values of x.

One very important continuous distribution that is used extensively in stochastic modelling is the *exponential distribution*. A random variable T is exponentially distributed with (rate) parameter $\lambda > 0$ if its distribution function is given by

$$F(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\lambda t}, & t \geq 0. \end{cases} \quad (1)$$

As we can imagine, the ubiquity of the exponential distribution in stochastic modelling is due to its simple formulation in terms of a single parameter.

Differentiating (1) with respect to t gives the *density function*

$$f(t) = \begin{cases} 0, & t < 0 \\ \lambda e^{-\lambda t}, & t \geq 0. \end{cases} \quad (2)$$

The expected value (or mean) of T is $\mathbb{E}(T) = \frac{1}{\lambda}$.

The simplicity of the exponential distribution is also due to the *memoryless property*. That is, for $s, t > 0$, $P(T > s + t | T > t) = P(T > s)$, see, for example Norris [5, pages 70–71] or Ross [6, pages 201–204]. The memoryless property enables many performance measures in stochastic models that use the exponential distribution to have simple expressions. Incidentally, the exponential distribution is the only continuous distribution that exhibits the memoryless property.

Of course, the main restriction on the exponential distribution is its formulation in terms of only one parameter. The diagram below shows a histogram of the length of stay of 4696 patients at the Royal Melbourne Hospital who were transferred from other hospitals. Fitted to this dataset is an exponential distribution with $\lambda = 0.1779$ (the reciprocal of the mean length of stay). As we can see it is not very good.

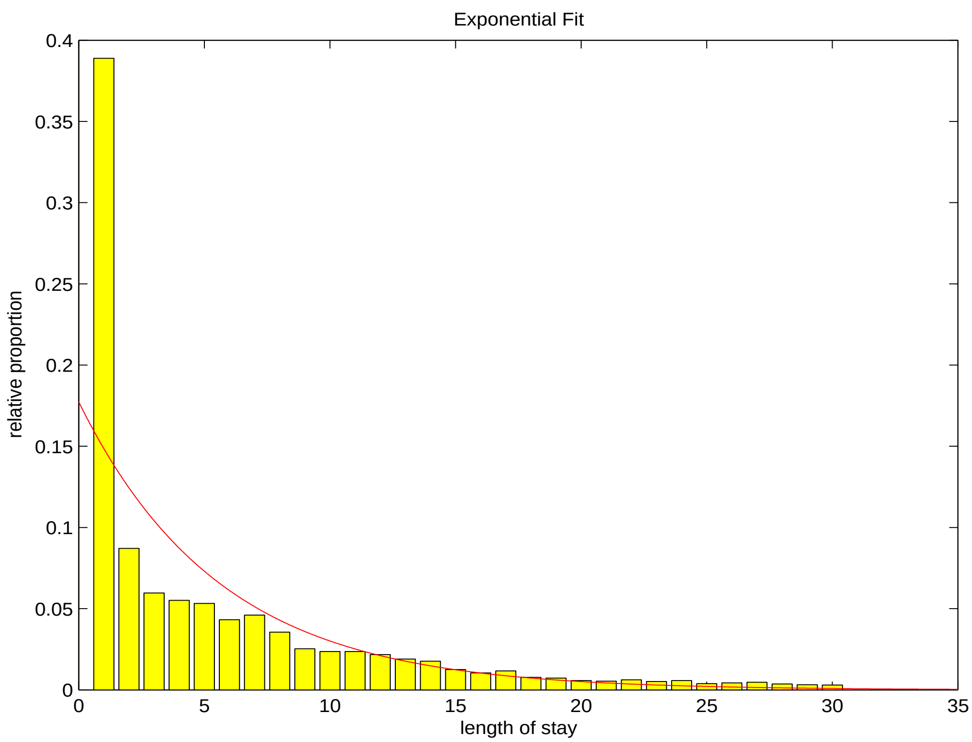

Figure 1: Exponential fit to the length of stay histogram.

So, in order to be able to fit datasets better, and hence develop better stochastic models, we need a class of distributions that are not only more versatile, but, hopefully, retain some of the exponential distribution's favourable properties. Here is where *PH* distributions come into the picture!

Before *PH* distributions are defined we need the concept of a *(finite-state) continuous-time Markov chain (CTMC)*, see Norris [5, Chapter 2] or Ross [6, Chapter 6]. Figure 2 shows the *state transition diagram* for a 4-state *CTMC*. The *state space* is $S = \{0, 1, 2, 3\}$. We select a state according to the *initial state probability vector* $(\alpha_0, \boldsymbol{\alpha}) = (\frac{1}{10}, \frac{1}{3}, \frac{2}{5}, \frac{1}{6})$. Once in a state we spend an exponentially distributed length of time there with parameter λ (for example, $\lambda = 12$ for state 1), and then move to another state with a certain rate (with rate 8 if we go from state 1 to 2, or rate 4 to state 0 - the 12 can be thought as the rate "out" of state 1). The process continues until we end up in state 0 where we stop. The matrix

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & -12 & 8 & 0 \\ 0 & 5 & -10 & 5 \\ 2 & 4 & 0 & -6 \end{pmatrix}, \tag{3}$$

which governs the transitions between states, is called the *infinitesimal generator*. The time from beginning to absorption is a random variable - a *PH* random variable.

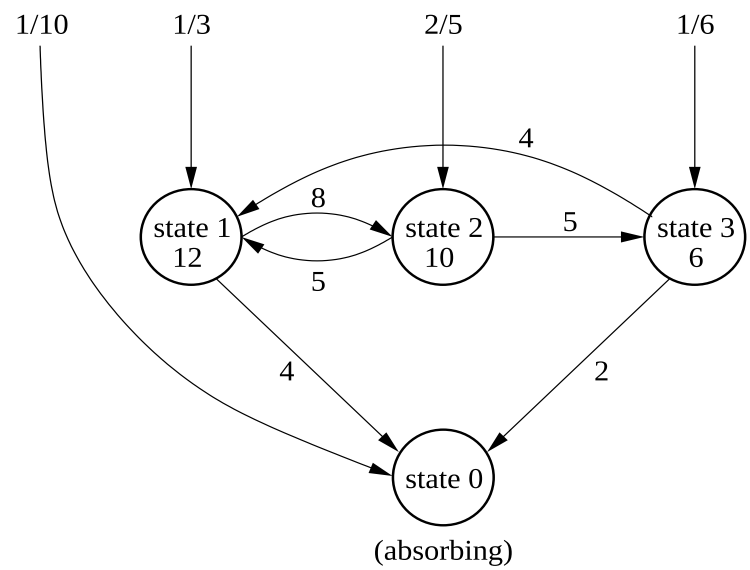

Figure 2: State transition diagram of a 4-state continuous-time Markov chain with one absorbing state.

Now for the mathematical definition. Consider a *CTMC* with $S = \{0, 1, 2, \ldots, p\}$ (state 0 is the absorbing state), $(\alpha_0, \boldsymbol{\alpha}) = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_p)$, (with $\sum_{i=1}^{p} \alpha_i = 1$), and infinitesimal generator $\boldsymbol{Q}$. The random variable that is the time to absorption is said to have a *(continuous) PH distribution*, see Neuts [4, Chapter 2]. For excellent introductions to *PH* distributions see Shaked and Shantikumar [7], Asmussen and Olsson [1], or Latouche and Ramaswami [3, Chapter 2].

The infinitesimal generator for the Markov chain can be written in block-matrix form as

$$\boldsymbol{Q} = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{t} & \boldsymbol{T} \end{pmatrix}. \tag{4}$$

Here, $\boldsymbol{0}$ is a $1 \times p$ vector of zeros. The $p \times p$ matrix $\boldsymbol{T}$ has negative diagonal entries and nonnegative off-diagonal entries such that the row sums are nonnegative with at least one row sum positive. The $p \times 1$ vector $\boldsymbol{t} = -\boldsymbol{T}\boldsymbol{e}$, where $\boldsymbol{e}$ is a $p \times 1$ vector of ones, is the absorption rate vector. We say that the *PH* distribution has a *order p representation* $(\boldsymbol{\alpha}, \boldsymbol{T})$. Typically, representations are not unique.

A *PH* distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{T})$ has distribution function, defined for $t \geq 0$, given by

$$F(u) = \begin{cases} \alpha_0, & t = 0 \\ 1 - \boldsymbol{\alpha} \exp(\boldsymbol{T}t)\boldsymbol{e}, & t > 0. \end{cases} \tag{5}$$

The corresponding density function, defined for $t > 0$, is

$$f(t) = -\boldsymbol{\alpha} \exp(\boldsymbol{T}t)\boldsymbol{T}\boldsymbol{e}. \tag{6}$$

Comparing equations (5) and (6) with equations (1) and (2), respectively, we can see that *PH* distributions are a "matrix analogue" of the exponential distribution. Some of the simplicity has been retained, but *PH* distributions are far more versatile because there are many more parameters ($p^2 + p - 1$ for an order p representation!).

In the next issue's article I will introduce Coxian distributions, a particular type of *PH* distribution that has been used extensively in healthcare modelling, and discuss how they have been used.

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